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3. Sol:

$$\lambda^3+\lambda^2+\lambda+1=0$$

$\Rightarrow \lambda_1=-1, \quad \lambda_2=i \quad \lambda_3=-i.$

i.e. the general sol of homogeneous equ is:

$$y_0(t)=ce^{-t}+a\cos t+b\sin t.$$

Since $g(t)=e^{-t}+4t$, set

$$Y(t):=\alpha te^{-t}+\beta t+\gamma$$

$Y'''+Y''+Y'+Y=e^{-t}+4t \Rightarrow \alpha=\frac{1}{2}, \ \beta=4, \ \gamma=-4.$

i.e. $y(t)=ce^{-t}+a\cos t+b\sin t+\frac{1}{2}te^{-t}+4(t-1)$ is the sol.

6. Sol:

$\lambda^4+2\lambda^2+\lambda=0 \Rightarrow \lambda_1=\lambda_2=i, \quad \lambda_3=\lambda_4=-i.$

i.e. the sol of homo equ is:

$$y_0(t)=a_0\cos t+a_1t\cos t+b_0\sin t+b_1t\sin t.$$

Since $g(t)=3+\cos 2t$, set $Y(t)=\alpha\cos 2t+\beta$

$Y^{(4)}+Y''+Y=3+\cos 2t \Rightarrow \alpha=\frac{1}{9}, \ \beta=3.$

i.e. $y(t)=a_0\cos t+a_1t\cos t+b_0\sin t+b_1t\sin t+\frac{1}{9}\cos 2t+3$ is the sol.

13. Sol:

$$\lambda^3 - 2\lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1.$$

i.e. $Y(t) = t^2 (\alpha e^t) + t(a_0 + a_1 t + a_2 t^2 + a_3 t^3)$.

17. Sol:

$$\lambda^4 - \lambda^3 - \lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = \lambda_4 = 1$$

i.e. $Y(t) = t(a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t)\sin t + (c_0 + c_1 t)\cos t$

18. Sol:

$$\lambda^4 + 2\lambda^3 + 2\lambda^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 0 \quad \lambda_3 = -1+i \quad \lambda_4 = -1-i.$$

i.e. $Y(t) = c_1 e^t + (c_2 t + c_3) e^{-t} + t a_0 e^{-t} \cos t + t b_0 e^{-t} \sin t$.

20. Sol:

$$\begin{aligned}(D-a)(D-b)f &= (D-a)(Df - bf) \\ &= D^2 f - bDf - aDf + abf \\ &= D^2 - (a+b)Df + abf\end{aligned}$$

$$\begin{aligned}(D-b)(D-a)f &= (D-b)(Df - af) \\ &= D^2 f - aDf - bDf + abf \\ &= D^2 - (a+b)Df + abf \\ &= (D-a)(D-b)f.\end{aligned}$$

21. Sol:

a> $(D-2)(3e^{2t}) = 6e^{2t} - 6e^{2t} = 0.$

$(D+1)^2(te^{-t}) = (D+1)(e^{-t} - te^{-t} + te^{-t})$

$= (D+1)(e^{-t}) = -e^{-t} + e^{-t} = 0.$

$\Rightarrow (D-2)(D+1)^2(3e^{2t} - te^{-t})$

$= (D+1)^2[(D-2)(3e^{2t})] - (D-2)[(D+1)^2(te^{-t})]$

$= (D+1)^2(0) - (D-2)(0) = 0.$

b>. ~~[illegible]~~ easy to see $(D-2)^4(D+1)^3 Y = 0.$

$\Rightarrow (\lambda-2)^4(\lambda+1)^3 = 0$ i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \ \lambda_5 = \lambda_6 = \lambda_7 = -1.$

i.e. $Y(t) = e^{2t}(a_0 + a_1 t + a_2 t^2 + a_3 t^3) + e^{-t}(b_0 + b_1 t + b_2 t^2).$

c> ✓

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2. Sol:

set $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \dot{y} = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$\ddot{y} = \sum_{k=0}^{\infty} k(k+1) a_{k+1} x^{k-1} = \sum_{k=1}^{\infty} k(k+1) a_{k+1} x^{k-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$\ddot{y} - x\dot{y} - y = 0 \Rightarrow (n+1)(n+2) a_{n+2} - n a_n - a_n = 0$

i.e. $a_{n+2} = \dfrac{a_n}{n+2}.$

set $a_0=1,\ a_1=0$, we get:

$$y_1(t) = x^0 + \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$$

set $a_0=0,\ a_1=1$, we get

$$y_2(t) = \frac{1}{1}x^1 + \frac{1}{1\cdot 3}x^3 + \frac{1}{1\cdot 3\cdot 5}x^5 + \frac{1}{1\cdot 3\cdot 5\cdot 7}x^7 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!!}$$

c> $$y_1' = \sum_{k=0}^{\infty} \frac{2k\, x^{2k-1}}{(2k)!!} = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-2)!!} = x\sum_{k=1}^{\infty} \frac{x^{2k-2}}{(2k-2)!!}$$

$$= x\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!!} = x y_1 .$$

$$y_2' = \sum_{k=0}^{\infty} \frac{(2k+1)x^{2k}}{(2k+1)!!} = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k-1)!!} = 1 + x\sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!!}$$

$$= 1 + x\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!} = 1 + x y_2 .$$

then $W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ x y_1 & 1+x y_2 \end{pmatrix} = y_1 \neq 0$ for $x \neq 0$.

i.e. y_1 & y_2 form a fundamental set of sol's.

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39. Sol:

$$L(x^r) = x^r\left(r^2 + (\alpha-1)r + \beta\right)$$

i.e. $r_1 + r_2 = \frac{1-\alpha}{2}$, $r_1 r_2 = \beta$

d> $\begin{cases} \alpha>1, & \beta\geq 0 \\ \alpha=1, & \beta>0 \end{cases}$

a> $\alpha<1,\ \beta>0$; b> $\begin{cases} \alpha<1, & \beta\geq 0 \\ \alpha=1, & \beta>0 \end{cases}$; c> $\alpha>1,\ \beta>0$, e> $\alpha=1.\ \beta>0$.

40. Sol:

<u>known</u> : $y_1(x) = x^{r_1}$ is a sol.

set $y_2(x) = u(x)\, x^{r_1}$

then $y_2'(x) = r_1 x^{r_1-1} u(x) + u'(x)\, x^{r_1}$

$$y_2''(x) = r_1(r_1-1)\, x^{r_1-2} u(x) + 2 r_1 x^{r_1-1} u'(x) + u''(x)\, x^{r_1}$$

$$x^2 y_2'' + \alpha x y_2' + \beta y_2$$

$$= r_1(r_1-1)\, x^{r_1} u(x) + 2 r_1 x^{r_1+1} u'(x) + u''(x)\, x^{r_1+2}$$

$$+ \alpha r_1 x^{r_1} u(x) + \alpha x^{r_1+1} u'(x) + \beta x^{r_1} u(x)$$

$$= \left(x^2 [x^{r_1}]'' + \alpha x [x^{r_1}]' + \beta x^{r_1}\right) u(x)$$

$$+ (2r_1+\alpha)\, x^{r_1+1} u'(x) + x^{r_1+2} u''(x).$$

$$= x^{r_1+1}\left[(2r_1+\alpha)\, u'(x) + x\, u''(x)\right] = 0$$

$$\Rightarrow x u''(x) = -(2r_1+\alpha)\, u'(x)$$

$$\Rightarrow \frac{u''}{u'} = -\frac{2r_1+\alpha}{x}$$

$$\Rightarrow u' = C x^{-(2r_1+\alpha)} = C x^{-1}$$

$$\Rightarrow u(x) = C \log x + C$$

i.e. $y_2(x) = x^{r_1} \log x$ is a sol. $(x>0)$.